Stopping criteria

As usual with iterative procedures, we need some stopping criteria to decide when to stop. Given a tolerance τ for example $\sim 10^{-3}$, or 10^{-4})

• **test on the iterates:** at each iteration check if

$$\frac{\|\underline{x}^{(k)} - \underline{x}^{(k-1)}\|}{\|\underline{x}^{(k-1)}\|} \leq \tau$$

for some norm of vectors;

• **test on the residual:** when the test on the iterates is satisfied, check if

$$\frac{\|\underline{r}^{(k)}\|}{\|\underline{b}\|} \leq \tau \qquad (\underline{r}^{(k)} := \underline{b} - A\underline{x}^{(k)} \text{ is the residual})$$

When both are satisfied, stop and take $\underline{x}^{(k)}$ as solution.

Pseudocode for splitting methods

$$
\begin{aligned}
M\underline{x}^{(k)} &= \underline{b} + N\underline{x}^{(k-1)} = \underline{b} + (M - A)\underline{x}^{(k-1)} \\
&= \underline{b} - A\underline{x}^{(k-1)} + Mx^{(k-1)}
\end{aligned}
$$

$$
\Longrightarrow \underline{x}^{(k)} = \underline{x}^{(k-1)} + M^{-1}\underline{r}^{(k-1)}
$$

M is usually referred as a *preconditioner*.

Splitting iterative method

Input: $A \in \mathbb{R}^{n\times n}$, $\underline{b} \in \mathbb{R}^n$, $\underline{x}^{(0)} \in \mathbb{R}^n$, $tol \in \mathbb{R}^+$, maxiter $\in \mathbb{N}$
Choose $M \in \mathbb{R}^{n\times n}$ and set $\underline{r}^{(0)} = \underline{b} - A\underline{x}^{(0)}$
for $k = 1, 2, \dots,$ maxiter:
    Solve $M\underline{p}^{(k-1)} = \underline{r}^{(k-1)}$
    $\underline{x}^{(k)} = \underline{x}^{(k-1)} + \underline{p}^{(k-1)}$
    $\underline{r}^{(k)} = \underline{b} - A\underline{x}^{(k)}$
    If Stopping criteria are satisfied exit the loop
end
Output: $\underline{x}^{(k)}$

Error analysis

Unfortunately, the fact that the residual is small does not guarantee that the error $\underline{x} - \underline{x}^{(k)}$ is small.

$$\underline{r}^{(k)} := \underline{b} - A\underline{x}^{(k)} = A\underline{x} - A\underline{x}^{(k)} \;\longrightarrow\; \underline{x} - \underline{x}^{(k)} = A^{-1}\underline{r}^{(k)}.$$

Taking the norm in both sides we have

$$\begin{aligned}\|\underline{x} - \underline{x}^{(k)}\| = \|A^{-1}\underline{r}^{(k)}\| &\leq |||A^{-1}|||\, \|\underline{r}^{(k)}\| \\ &\leq |||A^{-1}|||\, \frac{\|\underline{r}^{(k)}\|}{\|\underline{b}\|}\, \|A\underline{x}\| \leq |||A^{-1}|||\, |||A|||\, \|\underline{x}\| \frac{\|\underline{r}^{(k)}\|}{\|\underline{b}\|}.\end{aligned}$$

Then we obtain

$$\frac{\|\underline{x} - \underline{x}^{(k)}\|}{\|\underline{x}\|} \leq |||A^{-1}|||\, |||A||| \frac{\|\underline{r}^{(k)}\|}{\|\underline{b}\|}.$$

If the number $\kappa(A) := |||A^{-1}|||\, |||A|||$ is big there is no control on the error, no matter how small the residual is. $\kappa(A)$ is called "condition number of A", and if $\kappa(A) >> 1$ the matrix is said to be ill-conditioned.

Concept of conditioning

When dealing with ill-conditioned matrices, any numerical method (direct or iterative) might produce unsatisfactory results.

Roughly speaking, a problem is *well-conditioned* if “small” perturbations on the data determine “small” perturbations on the results.

To clarify the concept of *conditioning* of a problem, let us consider a generic problem: find u solution of

$$(P) \qquad\qquad F(u, d) = 0,$$

where d are the data, and F is the law relating u to d.

Concept of conditioning

More precisely, let u be the solution of the problem (P) $\qquad F(u,d)=0$ corresponding to data d, and let δd be a perturbation on the data. Denote by δu the corresponding perturbation on the solution u. Then, instead of solving (P) we are solving

$$(\widetilde{P}) \qquad\qquad F(u+\delta u, d+\delta d)=0.$$

Problem (P) is well-posed (or, the solution of the problem depends continuously on the data) if

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \text{ such that if } \|\delta d\| \le \delta, \text{ then } \|\delta u\| \le \varepsilon$$

for some norm.

If a problem is well-posed, its (relative) condition number is the smallest constant $\kappa > 0$ that satisfies

$$\frac{\|\delta u\|}{\|u\|} \le \kappa \frac{\|\delta d\|}{\|d\|}$$

Example

Back to linear systems, let us make a conditioning analysis in a simple case. Assume that the possible errors are only on the right-hand side (and not on the matrix). Let $\delta \underline{b}$ be the errors on $\underline{b}$, small (in percentage), that is, for ε small,

$$\frac{\|\delta \underline{b}\|}{\|\underline{b}\|} \leq \varepsilon$$

The solution of the system with matrix A and right-hand side $\underline{\widetilde{b}} = \underline{b} + \delta \underline{b}$ will be $\underline{\widetilde{x}} = \underline{x} + \delta \underline{x}$:

$$A\underline{x} = \underline{b} \ \rightarrow \ A(\underline{x} + \delta \underline{x}) = \underline{b} + \delta \underline{b},$$

subtracting

$$A\delta \underline{x} = \delta \underline{b}, \quad \text{giving} \quad \delta \underline{x} = A^{-1}\delta \underline{b}.$$

Proceeding as we did before we have

$$\|\delta \underline{x}\| = \|A^{-1}\delta \underline{b}\| \leq |||A^{-1}|||\|\delta \underline{b}\| = |||A^{-1}|||\frac{\|\delta \underline{b}\|}{\|\underline{b}\|}\|\underline{b}\|$$

$$= |||A^{-1}|||\frac{\|\delta \underline{b}\|}{\|\underline{b}\|}\|A\underline{x}\| \leq |||A^{-1}|||\frac{\|\delta \underline{b}\|}{\|\underline{b}\|}|||A|||\|\underline{x}\|$$

We found

$$\|\delta \underline{x}\| \leq |||A^{-1}||| \frac{\|\delta \underline{b}\|}{\|\underline{b}\|} |||A||| \|\underline{x}\|$$

$$\Longrightarrow \frac{\|\delta \underline{x}\|}{\|\underline{x}\|} \leq |||A^{-1}|||\, |||A|||\varepsilon = \kappa(A)\varepsilon$$

A simple example to understand how a big condition number might affect the results.

$$\begin{pmatrix} 10^6 & 10^{-12} \\ 0 & 10^{-6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10^6 \\ 10^{-6} \end{pmatrix}$$

Exact solution $x_2 = 1, x_1 \simeq 1; \kappa_\infty(A) \simeq 10^{12}$. Now perturb only the first component of the right-hand side by 10^{-6}, and then only the second component by 10^{-6}. In both cases $\|\delta \underline{b}\|_\infty / \|\underline{b}\|_\infty \leq 10^{-12}$. What happens to the solution?